

# Event-Triggered Adaptive Horizon Model Predictive Control for Perturbed Nonlinear Systems

Pengfei Li<sup>1</sup>, Tao Wang<sup>1</sup>, Yu Kang<sup>2</sup> and Yun-Bo Zhao<sup>3</sup>

**Abstract**—This paper proposes a new event-triggered adaptive horizon model predictive control for discrete-time nonlinear systems with additive disturbance. With the event-triggered control scheme, the optimization problem is solved only at triggering instant and the event is triggered if the difference between the actual state and the predicted state exceeds the triggering threshold. The triggering threshold depends on the prediction horizon and becomes larger as the state approaches the terminal constraint set. Therefore, larger triggering intervals can then be obtained. Finally, a numerical example shows the effectiveness of the proposed scheme.

## I. INTRODUCTION

The model predictive control (MPC) technique has demonstrated exceptional success in studying the multi-input multi-output systems and constrained systems. At each time step, a constrained optimal control problem (OCP) is solved online, the sequence of optimal control actions is obtained and then the first element is applied to the system. Such processes will be repeated at the next time step [1]. The conventional MPC usually has large computation burden, which precludes the wider applications in many “fast” systems, e.g., quadrotors, servo systems and unmanned vehicles. Hence, it is theoretically and practically significant to treat this problem.

The consumption of the computing resource is mainly due to the frequent solving of the OCP with large prediction horizon and system dimensionality. To reduce the computing time, a remote server or cloud computing system with strong processing capability, which exchanges sensing and control data with sensors and actuators through communication network, can be employed, forming a networked MPC scheme [2]. On the other hand, an adaptive prediction horizon MPC algorithm can be adopted to reduce the computing complexity [3]. Additionally, event-triggered MPC has received much attention, and the computation burden can be reduced as the MPC algorithm is performed only when the triggering condition is violated (the error between the actual state and

the predicted one exceeds the triggering threshold), see, e.g., [4]–[8]. The positive consequences of the event-triggered mechanism include the reduction of the frequency of solving MPC and the saving of the communication resource.

However, the event-triggered mechanisms proposed in the aforementioned works rely on a fixed conservative triggering threshold because of the conservative Lipschitz constant and large prediction horizon. Recently, a novel event-triggered mechanism is proposed for continuous-time system in [9], where the prediction horizon shrinks as the state approaches the terminal set and the triggering threshold is inversely correlated with the shrinking prediction horizon. A similar idea can also be found in [10] where the self-triggered scheme and the update of the prediction horizon are co-designed for discrete-time system. But considering the constraint that the triggering interval (the time interval between two consecutive triggering instants) is not more than the prediction horizon, the shrinking horizon also brings negative effect especially when the state is in the terminal set, the prediction horizon is 0, leading to the consecutive triggering and the necessity of extra countermeasure to address it. Wang *et al.* [11] propose a dual event-triggered mechanism, i.e., two distinct triggering conditions are employed for the case that the state is outside the terminal set and the case in the terminal set. Although the consecutive triggering has been treated, the fixed triggering threshold is small and conservative. Both the aforementioned two ideas have their advantages and disadvantages, therefore, design an event-triggered mechanism that combines these advantages to reduce the usage of the computing and communication resources is our main focus.

In this paper, we propose a novel event-triggered MPC scheme, where the predictive state and control sequences with constant sequence length are generated collectively by the adaptive horizon MPC algorithm and the auxiliary control law. The triggering occurs only if the difference between the estimated state and the actual state exceeds a threshold that relies on the variable prediction horizon. For the state outside the terminal set, the triggering condition inherits the advantages of the one presented in [9], i.e., the triggering threshold increases as the prediction horizon shrinks, and meanwhile the upper bound of the triggering interval is relaxed from the shrinking prediction horizon in [9], [10] to a fixed large horizon. When the state is in the terminal set, the triggering condition still works and the triggering threshold reaches its maximum as the prediction horizon becomes zero, then transmission frequency can be significantly reduced.

This paper is organized as follows. Section II describes the system. The event-triggered MPC scheme is designed in

\*This work was supported by the National Key Research and Development Program of China (No. 2018AAA0100800, No. 2018YFE0106800), the National Natural Science Foundation of China (61725304, 61673361 and 61673350), the Science and Technology Major Project of Anhui Province (912198698036). Authors also gratefully acknowledge supports from the Chinese Academy of Sciences, the Youth Top-notch Talent Support Program and the Youth Yangtze River Scholar. (*Corresponding author: Yu Kang.*)

<sup>1</sup>Pengfei Li and Tao Wang are with the Department of Automation, University of Science and Technology of China, Hefei, 230027, China {puffylee, wangtao}@mail.ustc.edu.cn

<sup>2</sup>Yu Kang is with the Department of Automation, State Key Laboratory of Fire Science and Institute of Advanced Technology, University of Science and Technology of China, Hefei, China. kangduyu@ustc.edu.cn

<sup>3</sup>Yun-Bo Zhao is with the College of Information Engineering, Zhejiang University of Technology, Hangzhou, China. ybzhao@ieee.org

Section III. Section IV verifies the proposed scheme by a simulation example. Section V concludes this paper.

## II. SYSTEM DESCRIPTION AND PRELIMINARIES

We consider an open-loop unstable discrete-time nonlinear system with additive perturbation:

$$x(k+1) = f(x(k), u(k)) + w(k), \quad k \geq k_0 \quad (1)$$

where  $x(k) \in \mathcal{R}^n$  is the state,  $u(k) \in \mathcal{R}^p$  is the control input,  $w(k) \in \mathcal{R}^m$  is the bounded disturbance and  $k_0$  represents the initial time. We assume the control input and disturbance satisfy the following constraints:

$$u(k) \in \mathcal{U} \subseteq \mathcal{R}^p, \quad w(k) \in \mathcal{W} \subseteq \mathcal{R}^m, \quad \forall k \geq k_0. \quad (2)$$

The constraint sets  $\mathcal{U}$  and  $\mathcal{W}$  are compact and contain origin as an interior point. Let  $\|w\|_P := \max_{w \in \mathcal{W}} \sqrt{w^T P w}$  be the upper bound of the disturbance.

For ease of exposition, we introduce the nominal system of system (1) by letting  $w(k) \equiv 0$ ,

$$\hat{x}(k+1) = f(\hat{x}(k), \hat{u}(k)), \quad k \geq k_0 \quad (3)$$

which satisfies  $f(0, 0) = 0$ , where  $\hat{u}(k) \in \mathcal{U}$ .

In the following, the widely used concepts on the MPC are recalled. Performing the MPC algorithm needs to solve an OCP online. Let  $N$  be the prediction horizon. Then we define the following MPC cost function with the optimization variables  $\hat{\mathbf{u}}_N(k) = \{\hat{u}(k|k), \dots, \hat{u}(k+N-1|k)\}$  as

$$V(x(k), \hat{\mathbf{u}}_N(k), N) = \sum_{i=0}^{N-1} l(\hat{x}(k+i|k), \hat{u}(k+i|k)) + F(\hat{x}(k+N|k)) \quad (4)$$

where  $l(x, u) = x^T Q x + u^T R u$  and  $F(x) = x^T P x$  are the stage cost and terminal cost, respectively. Here  $Q$ ,  $R$  and  $P$  are some positive definite weighting matrices. Then the OCP is defined as

$$\begin{aligned} & \min_{\hat{\mathbf{u}}_N(k)} V(x(k), \hat{\mathbf{u}}_N(k), N) \\ \text{s.t. } & \hat{x}(k+i|k) = f(\hat{x}(k+i-1|k), \hat{u}(k+i-1|k)) \\ & \hat{x}(k|k) = x(k) \\ & \hat{u}(k+i|k) \in \mathcal{U}, \quad i = 0, \dots, N-1 \\ & \hat{x}(k+N|k) \in \mathcal{X}_f \end{aligned} \quad (5)$$

where  $\mathcal{X}_f$  is the terminal state constraint set. The optimal predicted state sequence is denoted by  $\hat{\mathbf{x}}_N^*(k) = \{\hat{x}^*(k+1|k), \dots, \hat{x}^*(k+N|k)\}$ . Define a set  $\mathcal{X}(N) = \{x(k) | \exists \hat{\mathbf{u}}_N(k) \in \mathcal{U}^N : \hat{x}(k+N|k) \in \mathcal{X}_f\}$ , which denotes the set of states that can reach  $\mathcal{X}_f$  within  $N$  steps.

The following assumptions impose some restrictions on the system model, stage cost function, terminal cost function and terminal state constraint set, and are often adopted to guarantee stability and feasibility.

*Assumption 1:* The function  $f(x, u)$  satisfying  $f(0, 0) = 0$  is local Lipschitz continuous w.r.t.  $x$  in  $\mathcal{X}(N) \times \mathcal{U}$ , i.e.,

$$\|f(x, u) - f(y, u)\|_P \leq L_x \|x - y\|_P \quad (6)$$

where  $L_x$  is Lipschitz constant and  $P$  is weighted matrix.

The above Lipschitz continuity assumption guarantees the existence of a unique solution of system (1). Furthermore, since system (1) is open-loop unstable, we have  $L_x \geq 1$ . Moreover, when  $L_x < 1$ , we can simply set  $L_x = 1$ .

*Assumption 2* ([11]): The stage cost function  $l(x, u)$ , terminal cost function  $F(x)$ , terminal constraint set  $\mathcal{X}_f$  and the auxiliary control law  $\kappa(x)$  satisfy the following properties, for all  $x, y \in \mathcal{X}(N)$ ,  $u \in \mathcal{U}$ ,

- (1)  $\mathcal{X}_f \subset \mathcal{X}(N)$ ,  $0 \in \mathcal{X}_f$  and  $\mathcal{X}_f$  is a compact set;
- (2)  $\kappa(x) \in \mathcal{U}$ ,  $\forall x \in \mathcal{X}_f$ ;
- (3)  $f(x, \kappa(x)) \in \mathcal{X}_f$ ,  $\forall x \in \mathcal{X}_f$ ;
- (4)  $F(f(x, \kappa(x))) - F(x) \leq -l(x, \kappa(x))$ ,  $\forall x \in \mathcal{X}_f$ ;
- (5) There exists a compact set  $\Phi \subset \mathcal{X}_f$  and  $0 \in \Phi$  such that  $f(x, \kappa(x)) \in \Phi$  for all  $x \in \mathcal{X}_f$ ;

*Remark 1:* The properties (1)-(4) are fairly standard and are widely adopted in MPC literature [12]. Here, we mainly focus on the last property. The terminal set is taken as  $\mathcal{X}_f = \{x | \|x\|_P \leq \epsilon_f\}$ . For simplicity, we also define the set  $\Phi = \{x | \|x\|_P \leq \epsilon\}$  with  $\epsilon < \epsilon_f$ . In general, determining the  $\epsilon$  analytically for a nonlinear system is an intractable task. Instead, for practical purpose we will find  $\epsilon$  via simulations from a sampling of the states in  $\mathcal{X}_f$ . In addition, the related Lipschitz constants can also be estimated by such approach.

## III. MAIN RESULTS

In this section, the event-triggered adaptive horizon MPC is designed, the feasibility and stability are then analyzed.

### A. Event-triggered Adaptive Horizon MPC

The schematic block diagram of the event-triggered MPC scheme is depicted in Fig.1 and the overview is stated as follows. Transmissions of the states measured periodically by the sensor are determined by the event-triggered mechanism. At each triggering time, the remote controller generates the predictive control and state sequences simultaneously, and transmits them to the actuator and the event generator, respectively. The control signal applied to the plant is updated and provided by the actuator periodically. In the following, we design the MPC-based controller and then determine the event-triggered mechanism.

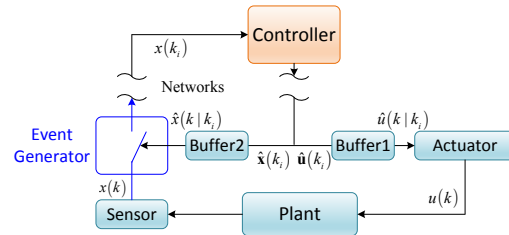


Fig. 1. Overall control structure, where two buffers provide the control input and predictive state from the corresponding sequences, respectively.

*1) MPC-based controller:* The role of the controller in Fig.1 is to generate predicted control and state sequences with length  $N_{\max}$  at each triggering instant. At time  $k_i$  (the  $i$ th triggering instant), both the state and control sequences consist of two parts, the first  $N(k_i)$  elements that are generated based on MPC with prediction horizon  $N(k_i)$ , and the

remaining  $N_{\max} - N(k_i)$  elements that are computed based on the auxiliary control law  $\kappa(x)$ , see Fig.2. To be specific, given the current state  $x(k_i)$ , the optimal predicted control sequence  $\hat{\mathbf{u}}_{N(k_i)}^*(k_i) = \{\hat{u}^*(k_i|k_i), \dots, \hat{u}^*(k_i + N(k_i) - 1|k_i)\}$  can be obtained by solving OCP (5) with prediction horizon  $N = N(k_i)$ . The corresponding predicted state sequence  $\hat{\mathbf{x}}_{N(k_i)}^*(k_i) = \{\hat{x}^*(k_i + 1|k_i), \dots, \hat{x}^*(k_i + N(k_i)|k_i)\}$  can also be obtained. According to this state sequence, the parameter  $N^*(k_i)$ , the minimum number of steps such that  $\hat{x}^*(k_i + N^*(k_i)|k_i) \in \mathcal{X}_f$ , can be determined.

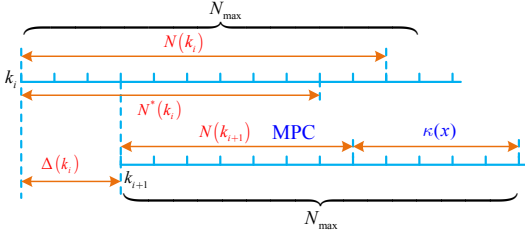


Fig. 2. A diagram of the relationships among  $N(k_i)$ ,  $N(k_{i+1})$ ,  $N^*(k_i)$ ,  $\Delta(k_i)$  and  $N_{\max}$ .

Denote the triggering interval between two consecutive triggering time instants by  $\Delta(k_i)$ , and  $\Delta(k_i) = k_{i+1} - k_i$ . The update of the prediction horizon is designed as:

$$N(k_{i+1}) = \max\{N^*(k_i) - \Delta(k_i) + 1, 0\} \quad (7)$$

where  $N(k_0) = N_0$  and  $N_0$  is the initial prediction horizon. The relationship among  $N(k_i)$ ,  $N(k_{i+1})$ ,  $N_{\max}$ ,  $N^*(k_i)$  and  $\Delta(k_i)$  is illustrated in Fig.2. One can easily observe that  $N(k_{i+1}) \leq N^*(k_i) \leq N(k_i)$ , i.e., the prediction horizon is non-increasing with respect to the triggering instant.

The procedure of computing the remaining  $N_{\max} - N(k_i)$  elements of state and control sequences is given as follows:

$$\begin{aligned} \hat{u}(k_i + j|k_i) &= \kappa(\hat{x}(k_i + j|k_i)); \\ \hat{x}(k_i + j + 1|k_i) &= f(\hat{x}(k_i + j|k_i), \hat{u}(k_i + j|k_i)) \end{aligned} \quad (8)$$

where  $j = N(k_i), \dots, N_{\max} - 1$  and  $\hat{x}(k_i + N(k_i)|k_i) = \hat{x}^*(k_i + N(k_i)|k_i)$ . It should be indicated that such procedure can be performed with high efficiency as some basic arithmetic operations are needed.

Incorporating the MPC results and the procedure (8), the state and control sequences with fixed length  $N_{\max}$  are given by  $\hat{\mathbf{x}}(k_i) = \{\hat{\mathbf{x}}_{N(k_i)}(k_i), \hat{x}(k_i + N(k_i) + 1|k_i), \dots, \hat{x}(k_i + N_{\max}|k_i)\}$  and  $\hat{\mathbf{u}}(k_i) = \{\hat{\mathbf{u}}_{N(k_i)}(k_i), \hat{u}(k_i + N(k_i)|k_i), \dots, \hat{u}(k_i + N_{\max} - 1|k_i)\}$ . We omit the superscript “\*” from  $\hat{\mathbf{x}}_{N(k_i)}^*(k_i)$  and  $\hat{\mathbf{u}}_{N(k_i)}^*(k_i)$  when the optimality of both sequences is not emphasized. Note that if  $N(k_i) = 0$ , only the procedure (8) is carried out.

2) *Event-triggered condition:* Due to the disturbance, the estimated error between the actual state and predicted one is growing larger over time. Hence, we design the following event-triggered condition

$$\|x(k) - \hat{x}(k|k_i)\|_P \leq (\epsilon_f - \epsilon)L_x^{-(N^*(k_i)+1)} - \frac{\|w\|_P}{L_x} \quad (9)$$

$$k - k_i \leq N_{\max}$$

where  $\|w\|_P \leq (\epsilon_f - \epsilon)L_x^{-N(k_0)-1}$  guarantees the validity of the first triggering condition. When either of the above conditions is violated at time  $k$ , the next triggering time is then  $k_{i+1} = k$ .

From the above triggering condition, one may notice that the following inequality holds

$$\begin{aligned} &\|x(k_{i+1} - 1) - \hat{x}(k_{i+1} - 1|k_i)\|_P \\ &\leq (\epsilon_f - \epsilon)L_x^{-(N^*(k_i)+1)} - \|w\|_P/L_x. \end{aligned}$$

Then, we further obtain

$$\begin{aligned} &\|x(k_{i+1}) - \hat{x}(k_{i+1}|k_i)\|_P \\ &\leq L_x \|x(k_{i+1} - 1) - \hat{x}(k_{i+1} - 1|k_i)\|_P + \|w\|_P \\ &\leq (\epsilon_f - \epsilon)L_x^{-N^*(k_i)}. \end{aligned} \quad (10)$$

This inequality provides an upper bound of the estimated error at the next triggering instant, and is a prerequisite for achieving the recursive feasibility.

*Remark 2:* Compared with the triggering condition proposed in [11], i.e.,

$$k_{i+1} = \begin{cases} \min_{k_i < k \leq k_i + N_{\max}} \{\|x(k) - \hat{x}(k|k_i)\|_P \geq c\} & \text{if } x(k) \notin \mathcal{X}_f; \\ \min_{k_i < k} \{\|x(k) - \hat{x}(k|k_i)\|_P \geq \alpha c\} & \text{otherwise.} \end{cases}$$

where the first triggering condition with a constant threshold  $c$  that is relatively small and leads to conservative triggering intervals, the incremental triggering threshold in (9) enables to extend the triggering interval, especially when the state is close to the terminal set (means small  $N^*(k_i)$ ). For the following triggering condition in [9] (the event-triggered adaptive horizon MPC for continuous-time nonlinear system is studied therein),

$$\begin{aligned} \|x(t_{k+1}) - \hat{x}^*(t_{k+1})\| &\leq c_1 e^{-L_x T_k^*} \\ t_{k+1} - t_k &\leq T_k^* \end{aligned}$$

where  $T_k^*$  plays a similar role of  $N^*(k_i)$ . Notice that the condition  $t_{k+1} - t_k \leq T_k^*$  hinders the merit brought by the incremental triggering threshold  $c_1 e^{-L_x T_k^*}$  because  $T_k^*$  (also an upper bound of the triggering interval) shrinks even to 0 (when  $x(k) \in \mathcal{X}_f$ ). This drawback has been overcome by the condition  $k - k_i \leq N_{\max}$  in (9), where  $N_{\max}$  is a large constant. In summary, the advantages of both triggering conditions have been maintained and the demerits have been bypassed due to the usage of the adaptive horizon MPC and the predicted state sequence with fixed length  $N_{\max}$ .

The overall event-triggered adaptive horizon MPC scheme is demonstrated by Algorithm 1.

## B. Recursive Feasibility

In this part, the feasibility is analyzed, which means the existence of the solution of MPC at each update time.

At each triggering time  $k_i$ , the state and control sequences are generated, where the first  $N(k_i)$  elements are obtained by solving the MPC and the last  $N_{\max} - N(k_i)$  element is obtained by (8). Note that if  $x(k_i) \in \mathcal{X}_f$ , i.e.,  $N(k_i) = 0$ ,

---

**Algorithm 1** Event-Triggered Adaptive Horizon MPC

---

**Initialization:** The initial state  $x(k_0)$ , the initial prediction horizon  $N_0$  and the length of predicted control sequence  $N_{\max}$ .

- 1: At any triggering instants  $k = k_i$ , initialize the state  $\hat{x}(k_i|k_i) = x(k_i)$ . If  $i > 0$ , then update the predicted horizon  $N(k_i)$  according to (7); else,  $N(k_0) = N_0$ .
  - 2: If  $N(k_i) = 0$ , go to step 3. Otherwise, solve the OCP to obtain the predicted state sequence  $\hat{\mathbf{x}}_{N(k_i)}^*(k_i)$  and control sequence  $\hat{\mathbf{u}}_{N(k_i)}^*(k_i)$ , and determine  $N^*(k_i)$ ;
  - 3: Compute the predicted state and control sequences with length  $N_{\max} - N(k_i)$  according to (8), and combine with  $\hat{\mathbf{x}}_{N(k_i)}^*(k_i)$  and  $\hat{\mathbf{u}}_{N(k_i)}^*(k_i)$  to get  $\hat{\mathbf{x}}(k_i)$  and  $\hat{\mathbf{u}}(k_i)$ .
  - 4: Apply  $\hat{u}(k|k_i)$  from  $\hat{\mathbf{u}}(k_i)$  to system (1), and update the time  $k = k + 1$ .
  - 5: If the triggering condition (9) is violated, set  $k_{i+1} = k$ ,  $i = i + 1$ , go to step 1. Otherwise, go to step 4.
- 

then both sequences are generated based on (8) and thus there is no need to perform the feasibility analysis. With triggering condition (9), prediction horizon update scheme (7) and the above analysis, the recursive feasibility of the adaptive horizon MPC can be established.

*Theorem 1:* Consider the constrained system (1) with the event-triggered condition (9). Suppose that the prediction horizon is determined according to (7), then the MPC is recursive feasible.

*Proof:* The result can be verified by induction. At time  $k_i$ , suppose that the MPC with prediction horizon  $N(k_i) > 0$  is feasible, and the resultant optimal control sequence is denoted by  $\hat{\mathbf{u}}_{N(k_i)}^*(k_i)$ . Then two cases,  $\Delta(k_i) \geq N^*(k_i) + 1$  and  $\Delta(k_i) < N^*(k_i) + 1$ , are considered.

For the case of  $\Delta(k_i) \geq N^*(k_i) + 1$ , then we have

$$\begin{aligned} \|x(k_{i+1})\|_P &\leq \|x(k_{i+1}) - \hat{x}(k_{i+1}|k_i)\|_P + \|\hat{x}(k_{i+1}|k_i)\|_P \\ &\leq (\epsilon_f - \epsilon)L_x^{-N^*(k_i)} + \epsilon \leq \epsilon_f \end{aligned} \quad (11)$$

This means the state  $x(k_{i+1})$  has been steered into  $\mathcal{X}_f$ , then the MPC will not be performed at time  $k_{i+1}$ .

For the case of  $\Delta(k_i) < N^*(k_i) + 1$ , the prediction horizon is given by  $N(k_{i+1}) = N^*(k_i) - \Delta(k_i) + 1$ . We construct the following control sequence  $\bar{\mathbf{u}}_{N(k_{i+1})}(k_{i+1})$  at triggering time  $k_{i+1}$  as follows:

$$\begin{aligned} &\bar{u}(k_{i+1} + j|k_{i+1}) \\ &= \begin{cases} \hat{u}^*(k_{i+1} + j|k_i), & j = 0, \dots, N^*(k_i) - \Delta(k_i) - 1 \\ \kappa(\hat{x}(k_{i+1} + N^*(k_i)|k_i)), & j = N^*(k_i) - \Delta(k_i) \end{cases} \end{aligned} \quad (12)$$

and use  $\bar{x}(k_{i+1} + j + 1|k_{i+1}) = f(\bar{x}(k_{i+1} + j|k_{i+1}), \bar{u}(k_{i+1} + j|k_{i+1}))$  with  $\bar{x}(k_{i+1}|k_{i+1}) = x(k_{i+1})$  to denote the predicted state by applying  $\bar{\mathbf{u}}_{N(k_{i+1})}(k_{i+1})$ .

In order to prove the recursive feasibility, we need to verify the satisfaction of the control constraint and terminal state constraint. To be specific,

- 1) The control constraint is met, i.e.,  $\bar{u}(k_{i+1} + j|k_{i+1}) \in \mathcal{U}$ ,  $\forall j = 0, \dots, N(k_{i+1})$ . This follows directly the feasibility of  $\hat{\mathbf{u}}_{N(k_i)}^*(k_i)$  and  $\kappa(x) \in \mathcal{U}$ ,  $\forall x \in \mathcal{X}_f$ .

- 2) By applying  $\bar{\mathbf{u}}_{N(k_{i+1})}(k_{i+1})$ , the terminal state  $\bar{x}(k_{i+1} + N(k_{i+1})|k_{i+1})$  is in the terminal set  $\mathcal{X}_f$ . Firstly, we study the difference between  $\bar{x}(k_{i+1} + N(k_{i+1})|k_{i+1})$  and  $\hat{x}^*(k_{i+1} + N(k_{i+1})|k_i)$ , and then based on which obtain the upper bound of the terminal state. Indeed, incorporating Assumption 1 and the construction of  $\bar{\mathbf{u}}_{N(k_{i+1})}(k_{i+1})$  yields

$$\begin{aligned} &\|\bar{x}(k_{i+1} + N(k_{i+1})|k_{i+1}) - \hat{x}^*(k_{i+1} + N(k_{i+1})|k_i)\|_P \\ &\leq L_x^{N(k_{i+1})} \|x(k_{i+1}) - \hat{x}^*(k_{i+1}|k_i)\|_P \\ &\leq L_x^{N^*(k_i) - \Delta(k_i) + 1} (\epsilon_f - \epsilon) L_x^{-N^*(k_i)} \\ &\leq \epsilon_f - \epsilon \end{aligned} \quad (13)$$

Then, by using the triangle inequality, we obtain

$$\begin{aligned} &\|\bar{x}(k_{i+1} + N(k_{i+1})|k_{i+1})\|_P \\ &\leq \|\hat{x}(k_i + N^*(k_i) + 1|k_i)\|_P + \epsilon_f - \epsilon \leq \epsilon_f \end{aligned} \quad (14)$$

where the first inequality holds because  $k_i + N^*(k_i) + 1 = k_{i+1} + N(k_{i+1})$ , and the second inequality holds because of the property (5) in Assumption 2 and the fact that  $\|\hat{x}(k_i + N^*(k_i)|k_i)\|_P \leq \epsilon_f$ .

The proof is then completed.  $\blacksquare$

### C. Stability analysis

In this part, the stability of the system with the designed event-triggered condition is analyzed. The line of the proof is similar to that of [9]. To be specific, we first show that the prediction horizon could reduce to zero in finite steps, and then show that the state will never leave  $\mathcal{X}_f$  forever.

According to (4), the optimal MPC value function at time  $k_i$  can be defined as

$$V_{\bar{N}}^*(x(k_i)) = V(x(k_i), \hat{\mathbf{u}}_{N(k_i)}^*(k_i), N(k_i)) \quad (15)$$

where  $\bar{N} = N(k_i)$ , and  $\hat{\mathbf{u}}_{N(k_i)}^*(k_i)$  is the optimal control sequence. In addition, since the MPC value function (4) is local Lipschitz continuous, then the optimal value function (15) is also Lipschitz continuous [12, Theorem C.29]. That is, there exists a constant  $L_V$  such that the following inequality holds for all  $x, y \in \mathcal{X}(\bar{N})$

$$|V_{\bar{N}}^*(x) - V_{\bar{N}}^*(y)| \leq L_V \|x - y\|_P \quad (16)$$

This constant can be approximated by simulation offline.

*Theorem 2:* Consider system (1), assume that the event-triggered adaptive horizon MPC scheme is implemented. If for any  $x(k_0) \in \mathcal{X}(N(k_0))/\mathcal{X}_f$  and  $w(k)$  satisfying

$$\|w\|_P \leq \min\{(\epsilon_f - \epsilon)L_x^{-N(k_0)}, (\theta_Q - \theta)/L_V\} \quad (17)$$

where  $L_V$  is defined in (16),  $\theta_Q = \min_{\|x\|_P > \epsilon_f} x^T Q x$  and  $\theta$  is a given small positive constant, then the prediction horizon reduces to zero within finite steps, and the system state  $x(k)$  remains in  $\mathcal{X}_f$  for all  $k \geq k_i$ , where  $k_i$  is the first triggering instant such that  $N(k_i) = 0$ .

*Proof:* We prove this result by contradiction. Suppose that  $N(k_i) > 0$  ( $x(k_i) \notin \mathcal{X}_f$ ) forever. Then we can claim that there exists an integer  $j > 0$  such that  $N(k_i) = N(k_j) > 0$  for all  $i \geq j$ . This is because the prediction horizon  $N(k_i)$

defined in (7) is nonincreasing with respect to  $i$ . Furthermore, we also observe that  $N^*(k_i) = N(k_i)$  and  $\Delta(k_i) = 1$  for all  $i \geq j$ . Now, we focus on the evolution of the optimal MPC value function after time  $k_j$ .

Let the optimal MPC value function (15) be the Lyapunov function, then the difference of Lyapunov functions at time  $k_{i+1}$  and  $k_i$  ( $i \geq j$  and  $\bar{N} = N(k_j)$ ) is given by

$$\begin{aligned} & V_{\bar{N}}^*(x(k_{i+1})) - V_{\bar{N}}^*(x(k_i)) \\ & \leq V_{\bar{N}}^*(x(k_{i+1})) - V_{\bar{N}}^*(\hat{x}^*(k_{i+1}|k_i)) \\ & \quad + V(\hat{x}^*(k_{i+1}|k_i), \bar{\mathbf{u}}_{\bar{N}}(k_{i+1}), \bar{N}) - V_{\bar{N}}^*(x(k_i)) \end{aligned} \quad (18)$$

By using the arguments in [13], we can directly obtain the following inequality

$$V(\hat{x}^*(k_{i+1}|k_i), \bar{\mathbf{u}}_{\bar{N}}(k_{i+1}), \bar{N}) - V_{\bar{N}}^*(x(k_i)) \leq -\|x(k_i)\|_Q^2 \quad (19)$$

where  $\bar{\mathbf{u}}_{\bar{N}}(k_{i+1})$  is defined in (12). Recall that

$$|V_{\bar{N}}^*(x(k_{i+1})) - V_{\bar{N}}^*(\hat{x}^*(k_{i+1}|k_i))| \leq L_V \|w\|_P \quad (20)$$

Substituting (19), (20) and (17) into (18) yields

$$V_{\bar{N}}^*(x(k_{i+1})) - V_{\bar{N}}^*(x(k_i)) \leq -\theta \quad (21)$$

Then, by induction we obtain  $V_{\bar{N}}^*(x(k_{i+1})) \leq V_{\bar{N}}^*(x(k_j)) - (i-j)\theta$ , which means that there always exists an instant  $t$  such that  $V_{\bar{N}}^*(x(k_t)) < 0$ , which contradicts with  $V^*(x) \geq 0$ . Therefore, the prediction horizon will reduce to 0 within finite steps.

Once  $N(k_i) = 0$ , then we have  $x(k_i) \in \mathcal{X}_f$ . According to property (5) in Assumption 2, we claim that  $\hat{x}(k|k_i) \in \Phi$  for all  $k_i < k \leq k_{i+1}$ . Note that

$$\|x(k)\|_P \leq \|x(k) - \hat{x}(k|k_i)\|_P + \|\hat{x}(k|k_i)\|_P \leq \epsilon_f \quad (22)$$

where the second inequality holds because of (10) with  $N^*(k_i) = 0$ . In fact, we also have  $N(k_{i+1}) = 0$ . Hence, the conclusion that  $x(k) \in \mathcal{X}_f$  for all  $k \geq k_i$  with  $N(k_i) = 0$  can be drawn by induction directly. These complete the proof. ■

*Remark 3:* The local Lipschitz constant of the optimal MPC value function is used in obtaining the decreasing property of the Lyapunov function. An alternative way is to use the Lipschitz constants of the system model, stage cost and terminal cost to obtain the difference between  $V(x(k_{i+1}), \bar{\mathbf{u}}_{\bar{N}}(k_{i+1}), \bar{N})$  and  $V_{\bar{N}}^*(x(k_i))$ , see, e.g. [11]. But such method is usually conservative due to the repeated use of these Lipschitz constants.

#### IV. NUMERICAL EXAMPLE

In this section, we consider the following example to show the effectiveness of the proposed event-triggered mechanism. The model of the continuous stirred tank reactor (CSTR) is adopted from [14]:

$$\begin{aligned} \frac{dC_A}{dt} &= \frac{q}{V}(C_{Af} - C_A) - k_0 \exp\left(-\frac{E}{RT}\right) C_A + w_{C_A} \\ \frac{dT}{dt} &= \frac{q}{V}(T_f - T) - \frac{\Delta H k_0}{\rho C_p} \exp\left(-\frac{E}{RT}\right) C_A \\ & \quad + \frac{UA}{V\rho C_p}(T_c - T) + w_T \end{aligned} \quad (23)$$

TABLE I  
PARAMETERS DESCRIPTION

Symbols	Value	Symbols	Value
$\rho$	1000 g/l	$C_p$	0.239 J/gK
$\Delta H$	$-5 \times 10^4$ J/mol	$E/R$	8750 K
$k_0$	$7.2 \times 10^{10}$ min $^{-1}$	$U.A$	$5.4 \times 10^4$ J/min K
$q$	100 l/min	$T_f$	350 K
$V$	100 l	$C_{Af}$	1.0 mol/l

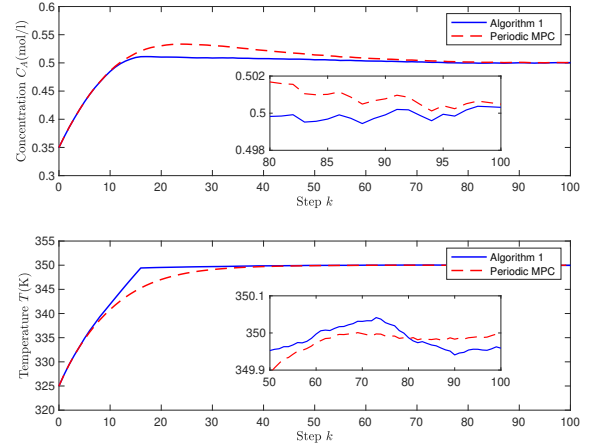


Fig. 3. State evolutions under Algorithm 1 and the periodic MPC.

where  $C_A$  is the concentration of  $A$  in the reactor,  $T$  and  $T_c$  are the temperatures of the reactor and the coolant stream, respectively. The related parameters of the model are given in Table. I. Under these conditions, an unstable equilibrium state is  $C_A^e = 0.5$  mol/l,  $T^e = 350$  K and  $T_c^e = 300$  K. The control input, i.e., the coolant stream, is constrained to  $280 \text{ K} \leq T_c \leq 370 \text{ K}$ . In order to obtain the nonlinear system (1), we define the system state as  $x = [C_A - C_A^e, T - T^e]^T$ , the control input as  $u = T_c - T_c^e$ , and then adopt the forward-Euler discretized method with sampling interval  $T_s = 0.03$  min. The initial condition is  $C_A(0) = 0.38$  mol/l,  $T(0) = 320$  K. Our aim is to design control scheme to achieve the equilibrium state.

The stage cost and terminal cost are given by  $l(x, u) = x^T Q x + u^T R u$  and  $F(x) = x^T P x$ , respectively, where  $Q = \text{diag}\{1, 1/1500\}$  and  $R = 1/900$ . The initial prediction horizon  $N(k_0) = 12$  and the fixed length of the predicted state and control sequences is  $N_{\max} = 30$ . The terminal state constraint set is  $\mathcal{X}_f = \{x : \|x\|_P \leq 1.4832\}$  and the auxiliary local controller is  $\kappa(x) = Kx$ , where

$$P = \begin{bmatrix} 116.5500 & 0.2245 \\ 0.2245 & 0.6671 \end{bmatrix}, \quad K = [-105.5642 \quad -3.0915]$$

A smaller set determined by property (5) in Assumption 2 is  $\Phi = \{x : \|x\|_P \leq 1.4384\}$ . The computed local Lipschitz constant is  $L_x = 1.3865$ . Let  $w = [w_{C_A}, w_T]$ , then according to Theorem 2, we assume the allowable disturbance set is  $\mathcal{W} = \{w \in \mathcal{R}^2 \mid \|w\|_P \leq \bar{w}\}$  where  $\bar{w} = 0.0667$ .

In the simulation, the comparisons of the proposed event-triggered adaptive horizon MPC scheme and the conventional periodic fixed-horizon MPC ( $N = 12$ ) in [13] are made and the results are shown in Fig.3-5. Note that the OCPs are



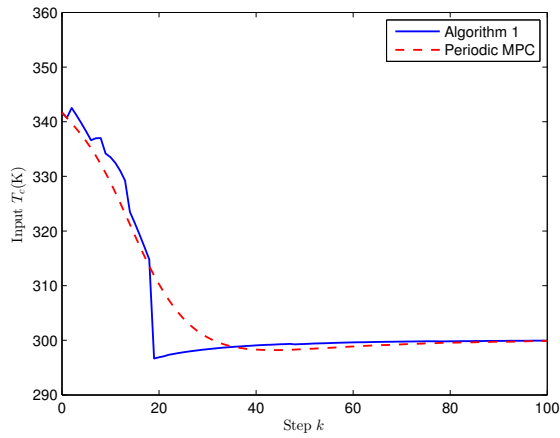


Fig. 4. Control inputs generated by Algorithm 1 and the periodic MPC.

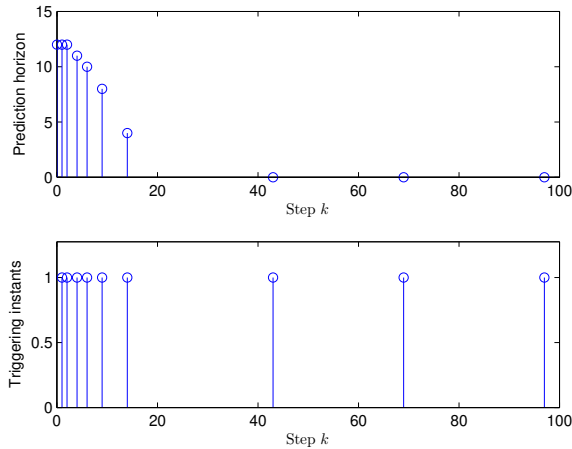


Fig. 5. The prediction horizon and the triggering instants obtained with  $\|w\|_P \leq 0.0667$ .

solved by MATLAB subroutine `fmincon`. Fig.3 shows the state evolutions and Fig.4 depicts the corresponding input trajectories. It can be observed that under both two control schemes the states converge to the equilibrium point fast and enter into the terminal set  $\mathcal{X}_f$  within finite steps. The actual control inputs also satisfy the control constraint. The updates of the prediction horizon and the triggering instants are depicted in Fig.5. Note that when the state is in terminal set ( $N(k_i) = 0$ ), the event-triggered mechanism in (9) is still valid. We can see that the triggering interval becomes larger as the prediction horizon shrinks and a larger triggering threshold can thus be obtained. In particular, the triggering intervals are relatively large when the state stays in the terminal set  $\mathcal{X}_f$ . The number of triggering is 13 among 100 time steps, which means 87% of communication resources are saved. The total computing time and control performance (defined by  $\sum_{k=0}^T l(x_k, u_k)$  with  $T$  being the running steps) are presented in Table.II. It can be seen that although the total computing time is significantly reduced, the degradation of the control performance is fairly small.

## V. CONCLUSION

The problem of the event-triggered MPC for a class of discrete-time nonlinear systems with additive disturbance

TABLE II  
COMPUTING TIME AND CONTROL COST

	Total computing time	Control cost
Periodic MPC	14.7645s	24.2923
Algorithm 1	2.3499s	26.6378

has been addressed. A new event-triggered adaptive horizon MPC with networked configuration has been designed, where the triggering condition depends on the prediction horizon. On the one hand, the OCP is solved only at triggering instant determined by the event-triggered mechanism; on the other hand, the prediction horizon is shrinking as the state is driven to the terminal set, which in turn enlarges the triggering interval and further reduces the number of triggering. Under this event-triggered MPC scheme, the computing time has been reduced, the computation and communication resources have been saved. Future work involves considering the delays and the packet dropouts in transmission over networks.

## REFERENCES

- [1] D. Q. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.
- [2] P. Li, Y. Kang, Y.-B. Zhao, and T. Wang, "Networked dual-mode adaptive horizon MPC for constrained nonlinear systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, 2020, doi: 10.1109/TSMC.2020.2971241.
- [3] D. W. Griffith, L. T. Biegler, and S. C. Patwardhan, "Robustly stable adaptive horizon nonlinear model predictive control," *J. Process Control*, vol. 70, pp. 109–122, 2018.
- [4] A. Eqtami, D. V. Dimarogonas, and K. J. Kyriakopoulos, "Novel event-triggered strategies for model predictive controllers," in *Proc. 50th IEEE Conf. Decision Control & Eur. Control Conf.*, Orlando, FL, USA, 2011, pp. 3392–3397.
- [5] H. Li and Y. Shi, "Event-triggered robust model predictive control of continuous-time nonlinear systems," *Automatica*, vol. 50, no. 5, pp. 1507–1513, 2014.
- [6] N. He and D. Shi, "Event-based robust sampled-data model predictive control: A non-monotonic lyapunov function approach," *IEEE Trans. Circuits Syst. I*, vol. 62, no. 10, pp. 2555–2564, Oct. 2015.
- [7] C. Liu, J. Gao, H. Li, and D. Xu, "Aperiodic robust model predictive control for constrained continuous-time nonlinear systems: An event-triggered approach," *IEEE Trans. Cybern.*, vol. 48, no. 5, pp. 1397–1405, May. 2018.
- [8] P. Li, Y. Kang, Y.-B. Zhao, and T. Wang, "A novel self-triggered MPC scheme for constrained input-affine nonlinear systems," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, doi:10.1109/TCSII.2020.2999408.
- [9] K. Hashimoto, S. Adachi, and D. V. Dimarogonas, "Event-triggered intermittent sampling for nonlinear model predictive control," *Automatica*, vol. 81, pp. 148–155, 2017.
- [10] Z. Sun, L. Dai, K. Liu, D. V. Dimarogonas, and Y. Xia, "Robust self-triggered MPC with adaptive prediction horizon for perturbed nonlinear systems," *IEEE Trans. Autom. Control*, vol. 64, no. 11, pp. 4780–4787, Nov. 2019.
- [11] M. Wang, J. Sun, and J. Chen, "Input-to-state stability of perturbed nonlinear systems with event-triggered receding horizon control scheme," *IEEE Trans. Ind. Electron.*, vol. 66, no. 8, pp. 6393–6403, Aug. 2019.
- [12] J. B. Rawlings, D. Q. Mayne, and M. Diehl, *Model Predictive Control: Theory, Computation, and Design*. USA: Nob Hill Publishing, 2017.
- [13] C. Rajhans, S. C. Patwardhan, and H. Pillai, "Discrete time formulation of quasi infinite horizon nonlinear model predictive control scheme with guaranteed stability," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 7181–7186, 2017.
- [14] D. L. Marruedo, J. Bravo, T. Alamo, and E. Camacho, "Robust MPC of constrained discrete-time nonlinear systems based on uncertain evolution sets: application to a CSTR model," in *Proc. IEEE Int. Conf. Control Appl.*, Scotland, U.K., 2002, pp. 657–662.